

n -torsion clean rings

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ABSTRACT. Let n be an arbitrary natural number. The class of (strongly) n -torsion clean rings is introduced and investigated. Abelian n -torsion clean rings are somewhat characterized and a complete characterization of strongly n -torsion clean rings is given in the case when n is odd. Some open questions are posed at the end.

1. Introduction

Everywhere in the text, all rings are assumed to be associative with unity. Our notations and notions are in agreement with those from [L]. For instance, for such a ring R , $U(R)$ denotes the group of units, $\text{Id}(R)$ the set of idempotents and $J(R)$ the Jacobson radical of R , respectively. Besides, the finite field with m elements will be denoted by \mathbb{F}_m , and $\mathbb{M}_k(R)$ will stand for the $k \times k$ matrix ring over R ; $k \in \mathbb{N}$. For an element u of a group G , $o(u)$ will denote the order of u . The symbol $\text{LCM}(n_1, \dots, n_k)$ will be reserved for the least common multiple of $n_1, \dots, n_k \in \mathbb{N}$.

We will say a nil ideal I of R is nil of index k if, for any $r \in I$, we have $r^k = 0$ and k is the minimal natural number with this property. Likewise, we will say that I is nil of bounded index if it is nil of index k , for some fixed k .

Let us recall that a ring R is said to be *clean* if, for every $r \in R$, there are $u \in U(R)$ and $e \in \text{Id}(R)$ with $r = e + u$. If, in addition, the commutativity condition $ue = eu$ is satisfied, the clean ring R is called *strongly clean*. These rings were introduced by Nicholson in [N1] and [N2]. Both clean rings and their various specializations or generalizations are intensively studied since then (see, for example, [BDZ], [C], [D], [DM], [Di], [M] and references within).

A decomposition $r = e + u$ of an element r in a ring R will be called *n -torsion clean decomposition* of r if $e \in \text{Id}(R)$ and $u \in U(R)$ is n -torsion, i.e. $u^n = 1$. We will say that such a decomposition of r is *strongly n -torsion clean*, if additionally e and u commute.

The aim of this article is to investigate in detail the following proper subclasses of (strongly) clean rings:

DEFINITION 1.1. A ring R is said to be (*strongly*) *n -torsion clean* if there is $n \in \mathbb{N}$ such that every element of R has a (strongly) n -torsion clean decomposition and n is the smallest possible natural number with the above property.

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It is easy to see that boolean rings are precisely the rings which are (strongly) 1-torsion clean. Thus the classes introduced above can be treated as natural generalizations of boolean rings.

Let us notice that in [D] the class of (*strongly*) *invo-clean* rings was investigated. In our terminology, (strongly) invo-clean rings are precisely rings which are either (strongly) 1-torsion clean or (strongly) 2-torsion clean.

It is clear that every clean ring having the unit group of bounded exponent s is n -torsion clean for some n with $1 \leq n \leq s$. We will see below that n has to divide s , but does not have to be equal to s . Let us also observe that a homomorphic image of an n -torsion clean ring is always m -torsion clean, for some $m \leq n$. However, it is not clear whether n is a multiple of m . Notice that finite rings are always clean, so they are n -torsion clean for suitable n and it would be of interest to compute n for some classes of finite rings; for instance, for matrix rings over finite fields.

In the present paper we mainly concentrate on the case of strongly n -torsion clean rings. Our work is organized as follows: The first short section is of introductory character and it contains some basic observations and examples. Strongly n -torsion clean rings are investigated in Section 2. In particular, it is shown in Theorem 3.4 that such rings have to satisfy a polynomial identity of degree $2n$ and that their Jacobson radical is nil of bounded index. Theorem 3.13 offers a description of such rings which are abelian. Surprisingly, when n is odd, strongly n -torsion clean rings have to be commutative. Their precise description is given in the subsequent Theorem 3.15. We close the paper with some open questions.

2. Preliminaries and Examples

We begin with the following simple but useful observation. Its proof is provided for the sake of completeness.

LEMMA 2.1. *Let R be a (strongly) n -torsion clean ring. Then there exists a finite number of elements $r_1, \dots, r_k \in R$ with (strongly) clean decompositions $r_i = e_i + u_i$, $1 \leq i \leq k$, such that $n = \text{LCM}(o(u_1), \dots, o(u_k))$. In particular:*

- (1) *When the group $U(R)$ has finite exponent s , then n divides s .*
- (2) *When R is commutative, then $U(R)$ contains an element of order n .*

PROOF. For $r \in R$, let us set

$$r_{\min} = \min\{o(u) \mid r = e + u \text{ is a (strongly) } n\text{-torsion clean decomposition of } r \text{ and } o(u) \text{ divides } n\}.$$

Then each r_{\min} divides n . Thus $\text{LCM}(r_{\min} \mid r \in R)$ exists and also divides n . Moreover, we can pick elements $r_1, \dots, r_k \in R$ such that $\text{LCM}(r_{\min} \mid r \in R) = \text{LCM}(r_{1\min}, \dots, r_{k\min})$. The minimality of n gives $\text{LCM}(r_{\min} \mid r \in R) = n$. This completes the proof of the main statement. Subsequently, (1) and (2) follow. \square

It is well known that $1 + J(R) \subseteq U(R)$. In the class of rings for which the equality holds, the notation of n -torsion clean rings boils down to rings R for which the unit group $U(R)$ is of finite exponent n . Indeed, we have:

PROPOSITION 2.2. *Let R be a ring and $n \in \mathbb{N}$. Then:*

- (1) *If $r \in J(R)$, then the unit $1 + r$ has exactly one clean decomposition.*

(2) Suppose $U(R) = 1 + J(R)$. Then the following two conditions are equivalent:

- (a) R is (strongly) n -torsion clean.
- (b) R is (strongly) clean and the group $U(R)$ is of finite exponent n .

Moreover, if one of the equivalent conditions holds, then $R/J(R)$ is a boolean ring.

PROOF. (1) Let $r \in J(R)$. Observe that if $1+r = e+u$ is a clean decomposition of $1+r$, then $1-e = u-r \in \text{Id}(R) \cap U(R) = \{1\}$, that is, $e = 0$. This implies that $1+r$ has the unique clean decomposition $r+1 = 0 + (1+r)$.

(2) Suppose R is (strongly) n -torsion clean and $u \in U(R) = 1 + J(R)$. Then, by (1), $u = 0+u$ is the only clean decomposition of u and $u^n = 1$ follows, i.e. $U(R)$ is of finite exponent $s \leq n$.

Conversely suppose that R is (strongly) clean and $U(R)$ is a group of finite exponent s . Then it is clear that R is n -torsion clean ring, for some $n \leq s$. This yields the equivalence (a) \Leftrightarrow (b).

Since units always lift modulo the Jacobson radical, we have $U(R/J(R)) = \{1\}$. If R is strongly n -torsion clean, then $R/J(R)$ is m -torsion clean for some $m \leq n$. The above yields that $m = 1$, i.e. $R/J(R)$ is a boolean ring. \square

Notice that the ring $T_m(\mathbb{F}_2)$ of all upper triangular $m \times m$ matrices over the field \mathbb{F}_2 is clean, its Jacobson radical J consists of all strictly upper triangular matrices and $U(T_m(\mathbb{F}_2)) = 1 + J$. Thus, with Proposition 2.2 at hand, we deduce:

EXAMPLE 2.3. Let $m \in \mathbb{N}$ and let k be the smallest nonnegative integer such that $m \leq 2^k$. Then the ring $T_m(\mathbb{F}_2)$ is (strongly) 2^k -torsion clean.

Recall that a ring R is uniquely clean if every element of R has a unique clean presentation. Such rings were characterized in [NZ] as those abelian rings R such that $R/J(R)$ is boolean (whence $U(R) = 1 + J(R)$) and idempotents lift modulo $J(R)$. Notice that, as idempotents always lift modulo nil ideals, every ring R such that $R/J(R)$ is boolean and $J(R)$ is nil must be clean. Therefore, the above proposition also gives the following corollary. Its second statement generalizes Example 2.3.

COROLLARY 2.4. (1) Let R be a uniquely clean ring. Then R is n -torsion clean if and only if $U(R)$ is of exponent n ;
 (2) Let R be a ring such that $R/J(R)$ is boolean and $J(R)$ is nil of bounded index. Then R is n -torsion clean, where n is the exponent of $U(R)$. Moreover, n is a power of 2.

PROOF. (1) being an immediate consequence of the preceding discussion, let the ring R be as in (2). Then R is a UU ring (i.e. all units are unipotent) and so [DL, Theorem 3.4 (2)] implies that $U(R)$ is a 2-group. Now the thesis is a simple consequence of Proposition 2.2. \square

Proposition 2.2 demonstrates that, from the point of view of n -torsion clean property, rings with $U(R) = 1 + J(R)$ are, in some sense, not too interesting. The situation when the ring is Jacobson semisimple and has non-trivial group of units is much more interesting. The next example is of such nature and it shows that a ring can be n -torsion clean with n strictly smaller than the exponent of the group $U(R)$.

EXAMPLE 2.5. Let $R = \mathbb{M}_2(\mathbb{F}_2)$. Then R is 2-torsion clean and strongly 6-torsion clean.

PROOF. The ring R is nil clean by virtue of [BCDM]. Since the index of nilpotence of elements of R is at most 2, [D, Corollary 2.11] implies that R is invo-clean which is not boolean, so that it is 2-torsion clean. The above can be also checked by direct computations. For instance, the unit $r = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ of order 3 has a 2-torsion clean decomposition $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (but it does not have strongly 2-torsion clean decomposition). It is also easy to make elementary computations showing that R is strongly 6-torsion clean. \square

Let us notice that the unit group of $\mathbb{M}_2(\mathbb{F}_2)$ is isomorphic to the symmetric group \mathcal{S}_3 .

PROPOSITION 2.6. *Let $m, k \in \mathbb{N}$ be such that $m \leq 2^k$. Then $\mathbb{M}_m(\mathbb{F}_2)$ is n -torsion clean for some natural $n \leq 2^k$.*

PROOF. Set $R = \mathbb{M}_m(\mathbb{F}_2)$. It is known (cf. [BCDM, Theorem 3]) that R is a nil-clean ring. Thus every element x of R can be written as $x = e + z$ with $e = e^2$ and $z^m = 0$, as the index of nilpotence of elements in R is bounded by m . Now we can write $x = (1 + e) + (1 + z)$, where $1 + e$ is an idempotent and $(1 + z)^{2^k} = 1 + z^{2^k} = 1$, as $m \leq 2^k$. This enables us to conclude that every element of R has 2^k -torsion clean decomposition. In particular, R is n -torsion clean for some $n \leq 2^k$. \square

With the help of this proposition, we derive:

EXAMPLE 2.7. The rings $\mathbb{M}_3(\mathbb{F}_2)$ and $\mathbb{M}_4(\mathbb{F}_2)$, in view of Proposition 2.6, are n -torsion clean for some $n \leq 2^2$. The rings are, however, not invo-clean by virtue of [D, Corollary 2.11], and so $n \in \{3, 4\}$.

The linear group $GL(3, \mathbb{F}_2)$ is the unit group of $\mathbb{M}_3(\mathbb{F}_2)$. The group is known to be simple of order 168 and exponent 84.

3. Strongly n -torsion clean rings

The following technical lemma is crucial for our further considerations.

LEMMA 3.1. *Suppose that R is a ring and the element $a \in R$ possesses strongly n -torsion clean decomposition. Then the equality $(a^n - 1)((a - 1)^n - 1) = 0$ holds.*

PROOF. Let $a = e + v$ be a strongly n -torsion clean decomposition of a . Since e, v commute and $v^n = 1$, we deduce:

$$a^n - 1 = (e + v)^n - 1 = \sum_{i=0}^n \binom{n}{i} e^i v^{n-i} - 1 = \sum_{i=1}^n \binom{n}{i} e^i v^{n-i} \in Re.$$

This implies that

$$(3.1) \quad (a^n - 1)(1 - e) = 0 \quad \text{and, consequently,} \quad a^n - 1 = (a^n - 1)e.$$

Using $ve = (a - 1)e$, we also have $1 = v^n = (a - e)^n = (a - 1)^n e - a^n e + a^n$. This yields

$$(3.2) \quad a^n - 1 = (a^n - (a - 1)^n)e.$$

Applying (3.2) and the second equation of (3.1), we get $((a-1)^n - 1)e = 0$. This equality and the first equation of (3.1) now give together that $(a^n - 1)((a-1)^n - 1) = (a^n - 1)((a-1)^n - 1)(e + (1-e)) = 0$, as desired. \square

The following assertion is central.

LEMMA 3.2. *Let $n \in \mathbb{N}$ and let R be a ring satisfying the identity $(x^n - 1)((x - 1)^n - 1) = 0$. Then:*

- (1) $\text{char}(R) := |1 \cdot \mathbb{Z}|$ is finite and $J(R)$ is a nil ideal;
- (2) If n is odd, then R is a reduced ring of characteristic 2 and $J(R) = 0$;
- (3) If R is an algebra over a field F , then either R is abelian (i.e. all idempotents of R are central) or $\text{char}(F)$ divides n .

PROOF. Set $\phi(x) = (x^n - 1)((x - 1)^n - 1) \in \mathbb{Z}[x]$.

(1) Substituting $x = 3 \cdot 1$ in the identity $\phi(x) = 0$, we see that there exists $0 \neq m \in \mathbb{Z}$ such that $1 \cdot m = 0$ in R . This shows that characteristic $\text{char}(R)$ of R is finite.

Let $r \in J(R)$. Assume that r is not nilpotent. Then the multiplicatively closed set $S = \{r^k \mid k \in \mathbb{N}\}$ does not contain 0. Let P be a maximal ideal of R in the class of all ideals having empty intersection with S . So, P is a prime ideal of R , the ring $\bar{R} = R/P$ satisfies the same identity as R does and $\bar{r} = r + P \in J(\bar{R})$. Moreover \bar{r} is not nilpotent, as $S \cap P = \emptyset$. Thus, eventually replacing R by \bar{R} , we may additionally assume that the ring R is prime. Then, its subring $1 \cdot \mathbb{Z} = F$ of R is a domain. By the first part of the proof, $1 \cdot \mathbb{Z}$ is finite, so it is a field. This means that the element $r \in J(R)$ is algebraic over the field F and, as such, has to be nilpotent (cf. [L, Proposition 4.18]). This contradicts the choice of r and shows that every element of $J(R)$ is nilpotent.

(2) Suppose n is odd. Then, substituting $x = 0$ in the identity $\phi(x) = 0$, we obtain $2 = 0$, i.e. $\text{char}(R) = 2$.

Let $r \in R$ be such that $r^2 = 0$. If $n = 1$, then the identity $\phi(x) = 0$ shows that $(r - 1)r = 0$ and $r = 0$ follows immediately, as $r - 1$ is invertible.

Suppose now that $n \geq 3$. Notice that $(r - 1)^2 = 1$. Thus, as n is odd, $(r - 1)^n = (r - 1)$. Therefore, $r = \phi(r) = 0$. This shows that R has no nonzero nilpotent elements, i.e. R is reduced. Then also $J(R) = 0$ as, by (1), $J(R)$ is a nil ideal.

(3) Suppose R is an algebra over a field F . By (1), $\text{char}(F) = p \neq 0$. If n is odd then, using (2), R is a reduced ring, so it is abelian. Suppose now that n is even and p does not divide n . Thus $1 \cdot n$ is invertible in R . Let $e = e^2, r \in R$. Substituting $x := er(1 - e)$ in the identity $\phi(x) = 0$ and using the fact that n is even, we obtain $0 = ((er(1 - e))^n - 1)((er(1 - e) - 1)^n - 1) = ner(1 - e)$ and thus the equality $er(1 - e) = 0$ follows. Similarly $(1 - e)re = 0$. The above shows that every idempotent e of R is central, provided that $\text{char}(R)$ does not divide n . This completes the proof of the lemma. \square

REMARK 3.3. In regard to point (2) stated above, a routine argument demonstrates that when R is an n -torsion clean ring and n is odd, then $J(R) = 0$ and $\text{char}(R) = 2$. Indeed, let $0 = f + v$ be an n -torsion clean decomposition of 0. Then $-f = (-f)^n = v^n = 1$ and $\text{char}(R) = 2$ follows. Now, Proposition 2.2(1) yields that $1 = (r - 1)^n = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} r^i$, for any $r \in J(R)$. As n is odd, this

equation gives $0 = rw$, where $w = \sum_{i=1}^n \binom{n}{i} (-1)^{n-i} r^{i-1} \in 1 + J(R)$ is invertible in R , i.e. $r = 0$, as required.

Now we are ready to establish the following theorem.

THEOREM 3.4. *Let $n \in \mathbb{N}$. Suppose R is a strongly n -torsion clean ring. Then:*

- (1) *R is a PI-ring satisfying the polynomial identity $(x^n - 1)((x - 1)^n - 1) = 0$;*
- (2) *R has finite characteristic $\text{char}(R) = |1 \cdot \mathbb{Z}|$;*
- (3) *$J(R)$ is a nil ideal of index smaller than $(\text{char}(R))^n$;*
- (4) *When n is odd, then R is a reduced ring of characteristic 2 and $J(R) = 0$;*
- (5) *If R is an algebra over a field F , then:*
 - (i) *$J(R)$ is a nil ideal of index bounded by n ;*
 - (ii) *either R is abelian (i.e. all idempotents of R are central) or $\text{char}(F)$ divides n .*

PROOF. The first statement is a direct consequence of Lemma 3.1. Notice that, in virtue of Lemma 3.2, for completing the proof it remains only to show that $J(R)$ is nil of index bounded as indicated in the theorem.

Let $r \in J(R)$. We claim that $r^{(\text{char}(R))^n} = 0$. By Lemma 3.2(1), r is nilpotent. Furthermore, Proposition 2.2 (1) shows that the unit $1 + r$ has exactly one clean presentation. Thus $(1 + r)^n = 1$ follows, as R is n -torsion clean. Therefore $r^n \in S = (1 \cdot \mathbb{Z})[r] = (1 \cdot \mathbb{Z})r^{n-1} + \dots + (1 \cdot \mathbb{Z})$. By (1), $1 \cdot \mathbb{Z}$ is a finite ring with $c := \text{char}(R)$ elements. Hence the ring S is finite and has at most c^n elements. As $r \in S$ is nilpotent, its index of nilpotence has to be smaller than $|S| \leq c^n$ (to argue this, just consider the set $A \subseteq S$ of all powers of the element r and show that $|A|$ is the nilpotence index of r). This gives (3).

Suppose now that R is an algebra over a field F . Then S defined as above is, in this case, a finite dimensional algebra over $\mathbb{F}_p = 1 \cdot \mathbb{Z} \subseteq F$ of dimension not bigger than n . The dimension argument applied to the sequence of subspaces $S \supseteq Sr \supseteq Sr^2 \supseteq \dots$ shows that $r^n = 0$, when $r \in S$ is nilpotent. This yields (5)(i) and completes the proof of the theorem. \square

It is an important open question (see [N2, Question 2] and [CDN]) whether strongly clean rings are Dedekind finite. Since PI rings are Dedekind finite, the above theorem gives immediately the following corollary:

COROLLARY 3.5. *Strongly n -torsion clean rings are Dedekind finite.*

COROLLARY 3.6. *Let R be a strongly n -torsion clean ring. If R is a finitely generated algebra over a central noetherian subring, then $J(R)$ is nilpotent.*

PROOF. By Theorem 3.4(1), R satisfies a monic polynomial identity and $J(R)$ is a nil ideal. Now the thesis is a direct consequence of [B, Theorem 2.5]. \square

The following example shows that, in general, Jacobson radical of strongly n -torsion clean rings does not have to be nilpotent.

EXAMPLE 3.7. Let $F = \mathbb{F}_{p^k}$ and $F[X]$ be the polynomial ring in infinitely many commuting indeterminates from the set X . Set $R = F[X]/I$, where I is the ideal of $F[X]$ generated by all elements x^p , $x \in X$. Then R is a local ring and its Jacobson radical is not nilpotent. Making use of Propositions 2.2(1) and 3.8(1), one can easily check that R is $p(p^k - 1)$ -torsion clean.

In Theorem 3.15, stated in the sequel, we will present a complete characterization of strongly n -torsion clean rings in the case when n is odd. For doing so, the following proposition, which gives a characterization of strongly n -torsion clean rings which are subdirect products of fields, is needed.

- PROPOSITION 3.8. (1) *Let F be a field. Then F is n -torsion clean if and only if F is finite and $n = |F| - 1$.*
- (2) *A product of fields $\mathbb{F}_{p_1^{k_1}} \times \dots \times \mathbb{F}_{p_t^{k_t}}$ is n -torsion clean, where n is equal to $\text{LCM}(p_1^{k_1} - 1, \dots, p_t^{k_t} - 1)$;*
- (3) *A product $\prod_{i \in I} F_i$ of fields is n -torsion clean if and only if all fields F_i , $i \in I$, are finite, $\text{LCM}(|F_i| - 1 \mid i \in I)$ exists and is equal to n ;*
- (4) *Let R be a subdirect product of fields F_i , $i \in I$. Then R is n -torsion clean if and only if $\prod_{i \in I} F_i$ is n -torsion clean.*

PROOF. (1) Notice that any finite field F is n -torsion clean for some divisor n of $|F| - 1$. On the other hand, if F is any field which is n -torsion clean then, by Theorem 3.4, every element of F is a root of the polynomial $(x^n - 1)((x - 1)^n - 1) \in F[x]$, so $|F|$ is finite and $|F| \leq 2n$. Suppose that F is a finite n -torsion clean field and let $|F| - 1 = l \cdot n$. By what we have just shown it follows that $l = \frac{|F| - 1}{n} \leq 2 - \frac{1}{n} < 2$ and so $l = 1$ holds, i.e. $n = |F| - 1$, as required.

(2) Let $T = \mathbb{F}_{p_1^{k_1}} \times \dots \times \mathbb{F}_{p_t^{k_t}}$ and let n be as defined in (2). Notice that $n = \max\{o(u) \mid u \in U(T)\}$ and the order of any $u \in U(T)$ divides n . Therefore, T is m -torsion clean for some $m \leq n$.

For showing that $n = m$, it suffices to show that $n_i = p_i^{k_i} - 1$ divides n , for any $1 \leq i \leq t$. Note that, by (1), $F_i = \mathbb{F}_{p_i^{k_i}}$ is n_i -torsion clean. Furthermore, using Lemma 2.1, we can pick elements $r_1, \dots, r_s \in T$ and their clean decompositions $r_j = e_j + u_j$, $1 \leq j \leq s$, such that $m = \text{LCM}(u_1, \dots, u_s)$. For a fixed $1 \leq i \leq t$ consider the set $\{\pi_i(r_1), \dots, \pi_i(r_s)\} \subseteq F_i$, where π_i denotes the canonical projection of R onto F_i . Then, for every $a \in F_i$, a can be presented as $e + u$ with $u^{z_i} = 1$, where $z_i = \text{LCM}(o(\pi_i(u_1)), \dots, o(\pi_i(u_s)))$. Thus $n_i \leq z_i$. As z_i is a LCM of orders of elements in a cyclic group $U(F_i)$ of order n_i , we also deduce that $z_i \leq n_i$, i.e. $z_i = n_i$. This implies that n_i divides m , for any $1 \leq i \leq t$, as desired.

(3) Suppose the product $\prod_{i \in I} F_i$ is n -torsion clean. Then every field F_i is a homomorphic image of $\prod_{i \in I} F_i$. Thus, owing to (1), each F_i is a finite field. If $\text{LCM}(|F_i| - 1 \mid i \in I)$ would not exist, then there would exist indexes $i_1, \dots, i_k \in I$ such that $m = \text{LCM}(|F_{i_1}| - 1, \dots, |F_{i_k}| - 1) > n$. However, in virtue of (2), $T = F_{i_1} \times \dots \times F_{i_k}$ is m -torsion clean and $m \leq n$, as T is a homomorphic image of R . Thus $\text{LCM}(|F_i| - 1 \mid i \in I)$ do exist and we can assume that $\text{LCM}(|F_i| - 1 \mid i \in I) = \text{LCM}(|F_{i_1}| - 1, \dots, |F_{i_k}| - 1) = m$. Then it is clear that $n \leq m$. Notice also that $m \leq n$, as T is a homomorphic image of R , i.e. $n = m$. This gives (3).

(4) Let R be a subdirect product of fields F_i , $i \in I$. Suppose R is m -torsion clean. For every $i \in I$, F_i is a homomorphic image of R so, with the aid of (1), the field F_i is n_i -torsion clean, where $n_i = |F_i| - 1$. We also have $n_i \leq m$. Therefore, $\text{LCM}(|F_i| - 1 \mid i \in I)$ exists and the statement (3) shows that $\prod_{i \in I} F_i$ is n -torsion clean, where $n = \text{LCM}(|F_i| - 1 \mid i \in I)$. In particular, the order of any unit of R divides n and thus $m \leq n$ follows.

Let us fix $i \in I$ and let $F = F_i$ with $s = n_i$. Then, any $a \in F$ can be presented as $a = e + u$ with $u^m = 1$. Let k_1, \dots, k_s be orders of units in such presentations of all elements of F . Then, by construction, $k = \text{LCM}(k_1, \dots, k_s)$ divides m and also

divides $s = |F| - 1$ (as s is equal to the order of the group $U(F)$). In particular $k \leq s$. On the other hand, appealing to (1), F is s -torsion clean and this forces that $s \leq k$. However, this shows that $k = s = n_i$ divides m . This means that, for any $i \in I$, $n_i = s$ divides n . Consequently, $n = \text{LCN}(n_i \mid i \in I)$ divides m . By the first part of the proof $m \leq n$, so $n = m$ really follows.

Suppose now that $\prod_{i \in I} F_i$ is n -torsion clean and R is a subdirect products of fields F_i , $i \in I$. To complete the proof, it is enough to show R is m -torsion clean for some m . The statement (3) implies that the group $U(R)$ is of finite exponent, say k is the exponent. Then, for any $r \in R$, $e = r^k$ is an idempotent, and $r = (1 - e) + ((e - 1) + r)$ is a clean decomposition of a with $((e - 1) + r)^k = 1$. This allows us to conclude that R is m -torsion clean, for some $m \leq k$, as required. \square

The following result, which is needed later in the text, is also of some independent interest. Before stating it, let us recall that idempotents lift modulo an ideal J of R if, for any $a \in R$ such that $a^2 - a \in J$, there exists an idempotent $e \in R$ such that $e - a \in J$. If the idempotent e is uniquely determined by the element a , then we say that idempotents lift uniquely modulo J .

It is known that idempotents lift modulo nil ideals, thus the following lemma applies when J is a nil ideal of a ring R .

LEMMA 3.9. *Let $J \subseteq J(R)$ be an ideal of R . Suppose that idempotents lift modulo J . Then the following conditions are equivalent:*

- (1) *R is an abelian ring;*
- (2) *R/J is an abelian ring and idempotents lift uniquely modulo J .*

PROOF. Let $\pi: R \rightarrow R/J$ denotes the canonical homomorphism.

(1) \Rightarrow (2). Suppose the ring R is abelian. Since idempotents lift modulo J , $\text{Id}(R/J) = \pi(\text{Id}(R))$. Thus the ring R/J is abelian, as R is such. Let $e, f \in \text{Id}(R)$ be such that $e - f \in J \subseteq J(R)$. Then, by [KLM, Corollary 11], e and f are conjugate in R , i.e. there exists $u \in U(R)$ such that $e = ufu^{-1}$. However, all idempotents of R are central, so $e = f$. This, together with the assumption that idempotents lift modulo J yield that idempotents lift uniquely modulo J .

(2) \Rightarrow (1). The commutator of elements $a, b \in R$ will be denoted by $[a, b] := ab - ba$. Suppose (2) holds and let $e \in \text{Id}(R)$, $r \in R$. Then $f = e + er(1 - e)$ is also an idempotent and $[f, e] = er(1 - e)$. By assumption R/J is abelian, so $\pi([f, e]) = 0$. This shows that $er(1 - e) \in J$. Since $\pi(e) = \pi(f)$ and, by assumption, idempotents lift uniquely modulo J , we obtain $e = f$, i.e. $er(1 - e) = 0$. Now, replacing e by $1 - e$, we also have $(1 - e)re = 0$, for any $r \in R$. This means that every idempotent e of R is central, i.e. R is abelian, as required. \square

We will need in the sequel the following direct application of [DL, Theorem 3.2.].

LEMMA 3.10. *Let R be a ring and $u \in R$. Suppose that $m := \text{char}(R)$ is finite and $J(R)$ is a nil ideal of index $s + 1$, where $s \geq 0$. If $u^t - 1 \in J(R)$, then $u^{tm^s} = 1$.*

PROOF. [DL, Theorem 3.2.] states that if R is a ring satisfying assumptions of the lemma, then $(1 - r)^{m^s} = 1$, for any $r \in J(R)$. Now, if $u^t - 1 \in J(R)$, then there exists $r \in J(R)$ such that $u^t = 1 - r$ and $u^{tm^s} = 1$ follows. \square

The above lemma gives immediately the following corollary:

COROLLARY 3.11. *Let R be a ring of such that $\text{char}(R)$ is finite and $J(R)$ is nil of bounded index. If the group $U(R/J(R))$ is of finite exponent, then so is $U(R)$. If additionally $R/J(R)$ is clean (so R is also clean, as units and idempotents lift modulo nil ideals), then R is n -torsion clean, for some $n \in \mathbb{N}$.*

COROLLARY 3.12. *Let R be a ring of finite characteristic and J a nil ideal of R of bounded index. Then the following conditions are equivalent:*

- (1) R is an n -torsion clean ring, for some $n \in \mathbb{N}$.
- (2) R/J is an t -torsion clean ring, for some $t \in \mathbb{N}$.

PROOF. Suppose R/J is an m -torsion clean ring, for some $m \in \mathbb{N}$. Let $r \in R$. Since units and idempotents lift modulo J we can find $e \in \text{Id}(R)$ and $u \in U(R)$ such that $\bar{r} = \bar{e} + \bar{u}$ is an t -torsion clean decomposition of \bar{r} in R/J , where \bar{r} denotes the natural image of r in R/J . By Lemma 3.10, $u = 1$, where $m = \text{char}(R)$ and $s + 1$ is the nil index of the ideal J . This implies that R is n -torsion clean, for some $n \leq tm^s$.

The reverse implication is clear. □

The following theorem offers a characterization of strongly n -torsion clean abelian rings (compare with Theorem 3.4).

THEOREM 3.13. *For a ring R , the following conditions are equivalent:*

- (1) *There exists $n \in \mathbb{N}$ such that R is an n -torsion clean abelian ring.*
- (2)
 - (a) $\text{char}(R)$ is finite;
 - (b) $J(R)$ is nil of bounded index;
 - (c) Idempotents lift uniquely modulo $J(R)$;
 - (d) $R/J(R)$ is a subdirect product of finite fields F_i , where i ranges over some index set I , such that $\text{LCM}(|F_i| - 1 \mid i \in I)$ exists.
- (3) *R is an abelian clean ring such that $U(R)$ is of finite exponent.*

PROOF. (1) \Rightarrow (2). Suppose R is an n -torsion clean abelian ring. Then, Theorem 3.4 guarantees that $\text{char}(R)$ is finite and $J(R)$ is nil of finite index. In particular, R has properties (a) and (b). Since $J(R)$ is a nil ideal, idempotents lift modulo $J(R)$ and, by Lemma 3.9, they lift uniquely, so (c) holds.

Finally, by Theorem 3.4(1), $R/J(R)$ satisfies the polynomial identity $\phi(x) = 0$, where $\phi(x) = (x^n - 1)((x - 1)^n - 1) \in \mathbb{Z}[x]$. Therefore, $R/J(R)$ is a subdirect product of primitive PI-rings, say $R/J(R)$ is a subdirect product of primitive rings $\{R_i\}_{i \in I}$, for some index set I . Let us fix $i \in I$. Then R_i , as a homomorphic image of R , also satisfies the identity $\phi(x) = 0$. Consequently, by the classical Kaplansky's theorem (cf. [R]), each R_i has to be a central simple algebra, finite dimensional over its center C . Notice that, as $\text{char}(R)$ is finite, C is a field of nonzero characteristic, say $\mathbb{F}_p \subseteq C$. Observe also that, by Lemma 3.9, R_i is an abelian ring. This implies that R_i has to be a division algebra over \mathbb{F}_p . It is known (cf. [F, Corollary from page 48]) that every division algebra which is algebraic over a finite field is necessarily commutative. In particular, R_i has to be a field. In fact, it is a finite field, as R_i is contained in the splitting field of $\phi(x) \in \mathbb{F}_p[x]$. The above shows that $R/J(R)$ is a subdirect product of finite fields. Moreover, $R/J(R)$ is also, as a homomorphic image of R , strongly n' -torsion clean, for some $n' \leq n$. Therefore, making use of Proposition 3.8, we see that R satisfies the property (d). This completes the proof of the implication.

(2) \Rightarrow (3). Suppose (2) holds. We know, by (d) and Proposition 3.8, that $R/J(R)$ is a clean ring with the unit group $U(R)$ of finite exponent. The property (b) guarantee that $J(R)$ is a nil ideal and Corollary 3.11 yields that R is a clean ring with the unit group $U(R)$ is of finite exponent. Finally, properties (d), (c) together with Lemma 3.9 imply that R is an abelian ring.

The implication (3) \Rightarrow (1) is obvious. \square

In parallel to Theorem 3.13, one can state the following:

THEOREM 3.14. *For a ring R , the following conditions are equivalent:*

- (1) R is strongly n -torsion clean, for some $n \in \mathbb{N}$.
- (2) R is strongly clean and $U(R)$ is of finite exponent.

PROOF. (1) \Rightarrow (2). Suppose R is strongly n -torsion. Then clearly R is strongly clean. Next, observe that Theorem 3.4 implies that R is a PI-ring satisfying an identity of degree $2n$ and $J(R)$ is a nil ideal of bounded index. Using similar arguments as in the proof of Theorem 3.13, one can see that the quotient $R/J(R)$ is a subdirect product of a matrix rings, say $R_i = M_{m_i}(F_i)$, over finite fields F_i . Notice that, as $\text{char}(R)$ is finite, the set of characteristics of fields from the set $\mathcal{F} = \{F_i \mid i \in I\}$ is finite and also the number of fields of a given characteristic p is finite, as every such field is contained in the splitting field of a given polynomial of degree $2n$. Thus there are only finitely many classes of isomorphic fields in the set \mathcal{F} . Moreover, by the classical Amitsur-Levitzki's theorem (cf. [R]), each m_i is not greater than n , as every R_i satisfies a polynomial identity of degree $2n$. Therefore, the unit group of the product $\prod_{i \in I} R_i$ is a group of finite exponent. By Theorem 3.4, $\text{char}(R)$ is finite and $J(R)$ is a nil ideal of bounded index. Now, we can apply Lemma 3.10 to obtain that the group $U(R)$ is of finite exponent.

The implication (2) \Rightarrow (1) is clear. \square

We now have at our disposal all the necessary information to present a satisfactory structural characterization of strongly n -torsion clean rings, for all odd n .

THEOREM 3.15. *Suppose $n \in \mathbb{N}$ is odd. For a ring R , the following conditions are equivalent:*

- (1) R is a strongly n -torsion clean ring;
- (2) There exist integers $k_1, \dots, k_t \geq 1$ such that $n = \text{LCM}(2^{k_1} - 1, \dots, 2^{k_t} - 1)$ and R is a subdirect product of copies of fields $\mathbb{F}_{2^{k_i}}$, $1 \leq i \leq t$;
- (3) R is a clean ring in which orders of all units are odd, bounded by n and there exists a unit of order n .

PROOF. (1) \Rightarrow (2). Suppose R is a strongly n -torsion clean ring. Then, by Theorem 3.4 (4), R is a reduced ring of characteristic 2 and $J(R) = 0$. Thus, as every reduced ring is abelian, we can apply Theorem 3.13 to obtain that R is a subdirect product of finite fields F_i of characteristic 2, where $i \in I$, for some index set I . Now, Proposition 3.8 completes the proof of the implication.

The reverse implication (2) \Rightarrow (1) is a direct consequence of Proposition 3.8. The implication (2) \Rightarrow (3) is a tautology.

(3) \Rightarrow (1). Let R be as in (3). Then, as $(-1)^2 = 1$ and R has no units of even order, $-1 = 1$, i.e., $\text{char}(R) = 2$. Let us observe that R has to be reduced. Indeed, if $r^2 = 0$ for some $r \in R$, then $(1 + r)^2 = 1 + r^2 = 1$. Using again the fact that R has no units of even order, we get $r = 0$. It is known that in a reduced ring

all idempotents are central. Moreover, by assumption, R is a clean ring and, as every unit of R is of finite order bounded by n , the ring must be strongly m -torsion clean, for some $m \leq n!$. Now, because orders of units are odd, m has to be odd (as $u^{2k} = 1$ yields $u^k = 1$, when $o(u)$ is odd). Furthermore, bearing in mind the equivalence of statements (1) and (2), we conclude that $n = m$, as required. \square

It is worth to mention certain slightly unexpected, non-trivial consequences of the above theorem. Namely, not every odd natural number n can serve as torsion degree of strongly n -torsion clean rings and, for odd, n -torsion clean rings are always commutative.

Notice that, because every finite ring is clean, Theorem 3.15 forces the following:

COROLLARY 3.16. *For a finite ring R the following conditions are equivalent:*

- (1) *R is strongly n -torsion clean for some odd n ;*
- (2) *R has no units of even order;*
- (3) *R is isomorphic to a finite direct product of fields of characteristic 2.*

PROOF. By the Chinese Remainder Theorem, any subdirect product of finite number of fields is isomorphic to a direct product of fields. Now, the corollary is a straightforward consequence Theorem 3.15. \square

We close the paper with some problems of interest.

QUESTION 1. The matrix ring $M_n(\mathbb{F}_{2^k})$ is always m -torsion clean for some m . Compute m in terms of n and k ; is $m = n$ if $k = 1$?

Recall that some basic observations related to the above problem can be found in Proposition 2.6 and Examples 2.5 and 2.7. In particular $M_2(\mathbb{F}_2)$ is 2-torsion clean and, when $n \in \{3, 4\}$ then $M_n(\mathbb{F}_2)$ is m -torsion clean, where $2 < m \leq 4$. Christian Lomp checked for us, with the help of SageMath, that $n = m$ in the above cases.

We have seen in Theorem 3.14 that strongly n -torsion clean rings have unit groups $U(R)$ of finite exponent. For odd n , by Theorem 3.15, $n = \exp(U(R))$. Example 2.5 shows also such equality in the case of the ring $M_2(\mathbb{F}_2)$. We were kindly informed by Pace Nielsen, that such equality also holds for $M_3(\mathbb{F}_2)$, i.e. $M_3(\mathbb{F}_2)$ is strongly 84-torsion clean. Notice also that Example 3.7 offers yet another instance of equality $n = \exp(U(R))$.

Thus we pose the following two questions.

QUESTION 2. Let R be a strongly n -torsion clean ring. Is it true that $n = \exp(U(R))$?

QUESTION 3. Let R be an n -torsion clean ring. Is then necessary $U(R)$ of finite exponent?

In regard to both questions, are clean rings with cyclic units also strongly clean? For odd n , strongly n -torsion clean rings were characterized in Theorem 3.15. Besides, Theorem 3.13 offers a description of strongly n -torsion clean rings with extra assumption that the considered rings are abelian. So, we come to

QUESTION 4. Characterize strongly n -torsion clean rings, for even $n \in \mathbb{N}$.

If R is not abelian, then Theorem 3.4 (5) and arguments used in the proof of Theorem 3.14 show that, modulo the Jacobson radical (which is nil of bounded

index), Question 4 essentially reduces to the investigation of matrix rings over finite fields of characteristic dividing n .

It is also worthwhile noticing that (strongly) 2-torsion clean rings were classified in [D] under the name (strongly) invo-clean rings by using another approach. In fact, R is strongly invo-clean if and only if $R \cong R_1 \times R_2$, where R_1 is a ring for which $R_1/J(R_1)$ is boolean with $z^2 = 2z$ for every $z \in J(R_1)$, and R_2 is a ring which can be embedded in a direct product of copies of the field \mathbb{F}_3 .

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References

- [B] A. Braun, *The nilpotency of the radical in a finitely generated PI ring*, J. Algebra **89** (1984), no. 2, 375–396, DOI 10.1016/0021-8693(84)90224-2. MR751151
- [BCDM] S. Breaz, G. Călugăreanu, P. Danchev, and T. Micu, *Nil-clean matrix rings*, Linear Algebra Appl. **439** (2013), no. 10, 3115–3119, DOI 10.1016/j.laa.2013.08.027. MR3116417
- [BDZ] S. Breaz, P. Danchev, and Y. Zhou, *Rings in which every element is either a sum or a difference of a nilpotent and an idempotent*, J. Algebra Appl. **15** (2016), no. 8, 1650148, 11, DOI 10.1142/S0219498816501486. MR3528770
- [C] H. Chen, *On uniquely clean rings*, Comm. Algebra **39** (2011), no. 1, 189–198, DOI 10.1080/00927870903451959. MR2770888
- [CDN] V. Camillo, T. J. Dorsey, and P. P. Nielsen, *Dedekind-finite strongly clean rings*, Comm. Algebra **42** (2014), no. 4, 1619–1629, DOI 10.1080/00927872.2012.746976. MR3169656
- [D] P. V. Danchev, *Invo-clean unital rings*, Commun. Korean Math. Soc. **32** (2017), no. 1, 19–27, DOI 10.4134/CKMS.c160054. MR3608475
- [DL] P. V. Danchev and T.-Y. Lam, *Rings with unipotent units*, Publ. Math. Debrecen **88** (2016), no. 3-4, 449–466, DOI 10.5486/PMD.2016.7405. MR3491753
- [DM] P. V. Danchev and W. Wm. McGovern, *Commutative weakly nil clean unital rings*, J. Algebra **425** (2015), 410–422, DOI 10.1016/j.jalgebra.2014.12.003. MR3295991
- [Di] A. J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211, DOI 10.1016/j.jalgebra.2013.02.020. MR3037975
- [F] C. Faith, *Algebraic division ring extensions*, Proc. Amer. Math. Soc. **11** (1960), 43–53, DOI 10.2307/2032714. MR0111775
- [KLM] P. Kanwar, A. Leroy, and J. Matczuk, *Idempotents in ring extensions*, J. Algebra **389** (2013), 128–136, DOI 10.1016/j.jalgebra.2013.05.010. MR3065996
- [L] T. Y. Lam, *A first course in noncommutative rings*, 2nd ed., Graduate Texts in Mathematics, vol. 131, Springer-Verlag, New York, 2001. MR1838439
- [M] J. Matczuk, *Conjugate (nil) clean rings and Köthe’s problem*, J. Algebra Appl. **16** (2017), no. 4, 1750073, 14, DOI 10.1142/S0219498817500736. MR3635122
- [N1] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278, DOI 10.2307/1998510. MR0439876
- [N2] W. K. Nicholson, *Strongly clean rings and Fitting’s lemma*, Comm. Algebra **27** (1999), no. 8, 3583–3592, DOI 10.1080/00927879908826649. MR1699586
- [NZ] W. K. Nicholson and Y. Zhou, *Rings in which elements are uniquely the sum of an idempotent and a unit*, Glasg. Math. J. **46** (2004), no. 2, 227–236, DOI 10.1017/S0017089504001727. MR2062606
- [R] L. H. Rowen, *Polynomial identities in ring theory*, Pure and Applied Mathematics, vol. 84, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. MR576061

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